Symbolic Dynamics of "low complexity systems"

(TALK 1)

→ Background: In topological dynamics we consider a compact metric space X (metric d) and a continuous transformation

$$T: X \longrightarrow X$$

This couple "(X, T)" topological dynamical system. Here we have orbits:

$$\boxed{orb_T^+(x) = \{T^n(x) \mid n \geq 0\}} \qquad \forall x \in X.$$

Typically we will consider T to be a homeomorphism (bijective); in this case we have:

$$\boxed{orb_T(x) = \{T^n(x) \mid n \in \mathbb{Z}\}, \forall x \in X}$$

To classify a t.d.s. (X,T) we have properties:
(homeomorphism)

① **minimality**: $\forall x \in X$, $\overline{\mathrm{orb}_T(x)} = X$

$\iff$ U an open set in X, $\exists N \geq 0$, s.t.

$$X = \bigcup_{n=0}^{N} T^{-n} U$$

$\Downarrow$

② **transitivity**: ⊙ $\exists x \in X$, $\overline{\mathrm{orb}_T(x)} = X$ $\Leftarrow$

(when this is true prove that $\exists$ G_δ dense set of points like here)

⊙ $\forall U, V$ open sets, $\neq \emptyset$, $\exists n > 0$,

$$U \cap T^{-n} V \neq \emptyset$$

③ **weak-mixing**: $(X \times X, T \times T)$ is transitive.

"$(x,y) \rightsquigarrow T\times T(x,y) = (Tx, Ty)$"

(Ex: if (X,T) is w.m. $\Rightarrow$ $(\underbrace{X\times \cdots \times X}_{l \text{ times}}, T\times\cdots\times T)$ is transitive)

$\forall U, V, \tilde{U}, \tilde{V}$ open sets, $\neq \emptyset$, in X: $\exists n > 0$,

$U \cap T^{-n} V \neq \emptyset$, $\tilde{U} \cap T^{-n} \tilde{V} \neq \emptyset$

④ **mixing**: $\forall U, V$ open sets, $\neq \emptyset$,

$\exists N > 0$, $\forall n \geq N$,

$$\boxed{U \cap T^{-n} V \neq \emptyset}$$

In the minimal case to be w.mixing means that we cannot have "continuous eigenvalues":

$\lambda \in \mathbb{C}$, $|\lambda| = 1$, is a continuous eigenvalue for (X, T)

iff $\exists f \in C_{\mathbb{C}}(X)$, $\underbrace{f \circ T}_{U_T(f)} = \lambda \cdot f$

koopman operator $\to U_T(f)$

—— o ——

(5) <u>Invariant measures</u>: in X we can consider the σ-algebra of Borel sets $\mathcal{B}_X$ and look for probability measures: $\boxed{\mu : \mathcal{B}_X \to [0,1]}$

s.t.: $\forall B \in \mathcal{B}_X$, $T\mu(B) \doteq \underbrace{\mu(T^{-1}(B)) = \mu(B)}_{\text{invariant prob. measure.}}$

Theorem: given (X, T) a top. dynamical system there exists $\mu: \mathcal{B}_X \to [0,1]$ prob. measure s.t. $T\mu = \mu$.

Proof: ⊙ You need to prove that

$$\int_X f \, dT\mu = \int_X f \, d\mu$$

$\forall f \in C(X)$.

(Ex: $\int_X f \circ T \, d\mu = \int_X f \, dT\mu$)

⊙ First we need to construct such a μ:

start with $\nu: \mathcal{B}_X \to [0,1]$ any prob. measure ($\nu = \delta_x$, $x \in X$) and consider:

$$\mu_N = \frac{1}{N}\sum_{n=0}^{N-1} T^n_* \nu \xrightarrow[\text{up to a subsequence } (N_i)]{} \mu \Leftarrow \text{candidate.}$$

$$T^n_*\nu = \nu(T^{-n}(\cdot))$$

[$\mathcal{M}(X)$ space of prob. measures in $\mathcal{B}_X$ is compact-metric]
weak-* topology.

→ let $f \in C(X)$:

$$\int_X f\circ T \, d\mu = \lim_{N\to\infty} \int_X f\circ T \, d\left(\frac{1}{N}\sum_{n=0}^{N-1} T^n_*\nu\right)$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X f\circ T^{n+1} d\nu$$

$$= \lim_{N\to\infty}\left(\cancel{\frac{\int f\circ T^N d\nu}{N}} - \frac{\int f\, d\nu}{N} + \frac{1}{N}\sum_{n=0}^{N-1}\int_X f\circ T^n d\nu\right)$$

$$= \lim_{n \to \infty} \int_X f \, d\mu_{x_n} = \int_X f \, d\mu \qquad \blacksquare$$

<u>Conclusion:</u> given (X, T) the space

$$\mathcal{M}_T(X) = \{\mu : \mathcal{B}_X \to [0,1] \mid \mu \text{ is a prob. m.}, \; T\mu = \mu\}$$

$$\neq \emptyset.$$

[when $\mathcal{M}_T(X)$ is a singleton we say that (X, T) uniquely ergodic]

We say $\mu \in \mathcal{M}_T(X)$ is ergodic iff.

$$\forall B \in \mathcal{B}_X, \; \mu(B \,\Delta\, T^{-1}B) = 0 \Rightarrow \mu(B) = 0 \text{ or } 1.$$

$$(B =_\mu T^{-1}B)$$

6 Factors & extensions :

Given (X,T) and (Y,S) top. dyn. systems we say (Y,S) is a factor of (X,T) or that (X,T) is an extension of (Y,S) iff.

$\exists \quad \pi : X \longrightarrow Y$ continuous, onto

s.t. the diagram commutes.

$$
\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowright"] & X \arrow[d, "\pi"] \\
Y \arrow[r, "S"'] & Y
\end{tikzcd}
$$

$$\boxed{\pi \circ T = S \circ \pi}$$

Try to classify top. dyn. syst. by the factors ___

(X, T) and (Y, S) are conjugated if π in previous slide is bijective. (systems "are equal").-

<u>Ex</u>: Assume there is $f: X \to S^1$ continuous function and $\lambda \in S^1$ ($S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$)

s.t. $f \circ T = \lambda f$. We can write:

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ S^1 \supseteq f(X) \arrow[r, "S"] & f(X) \subseteq S^1 \end{tikzcd}$$

rotation factor

$$\underline{\underline{S(f(x))}} = \lambda f(x) \underset{\uparrow \text{know}}{=} \underline{\underline{f \circ T}}$$

If: $\lambda = \exp(2\pi i \alpha)$, $\alpha \in \mathbb{Q}^c \Rightarrow f(X) = S^1$.

II Symbolic Dynamics: (X will be particular and T too)

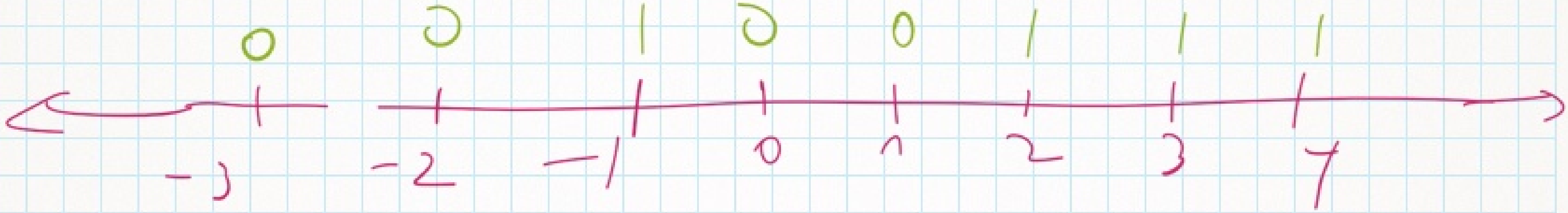
Let A be a finite set that we also call alphabet

Consider:

$$X = A^{\mathbb{Z}} = \{x = (x_i)_{i \in \mathbb{Z}} \mid \forall i \in \mathbb{Z},\ x_i \in A\}$$ ← fullshift on A.

Ex: $A = \{0,1\}$, $\{0,1\}^{\mathbb{Z}}$ ← bi-infinite sequences of 0's and 1's.

We write:

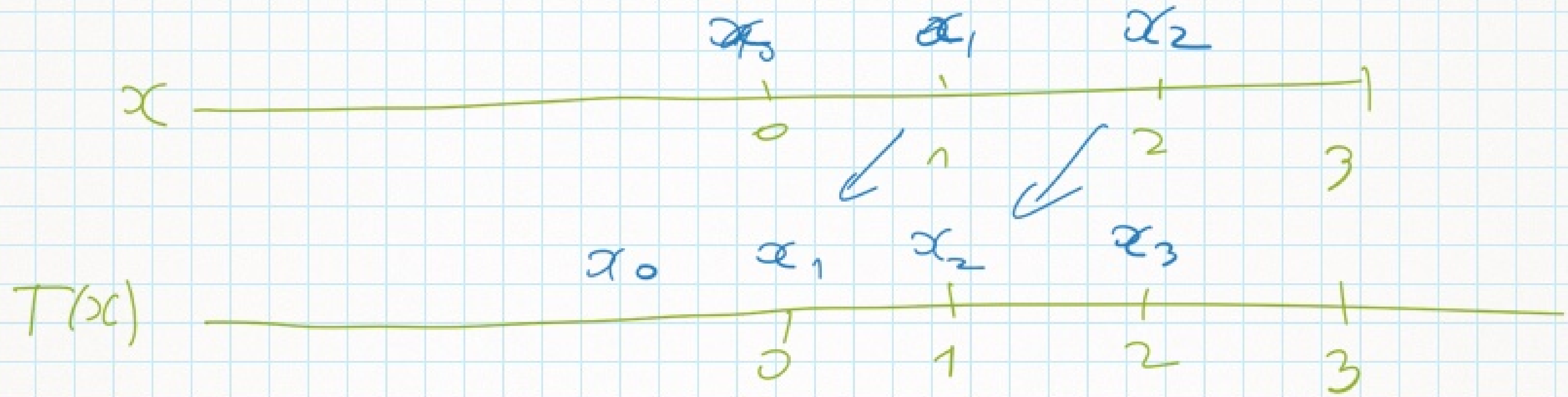


→ I:

$$T: A^{\mathbb{Z}} \longrightarrow A^{\mathbb{Z}}$$

$$x = (x_i)_{i \in \mathbb{Z}} \mapsto T(x) = (x_{i+1})_{i \in \mathbb{Z}}$$

(left) shift



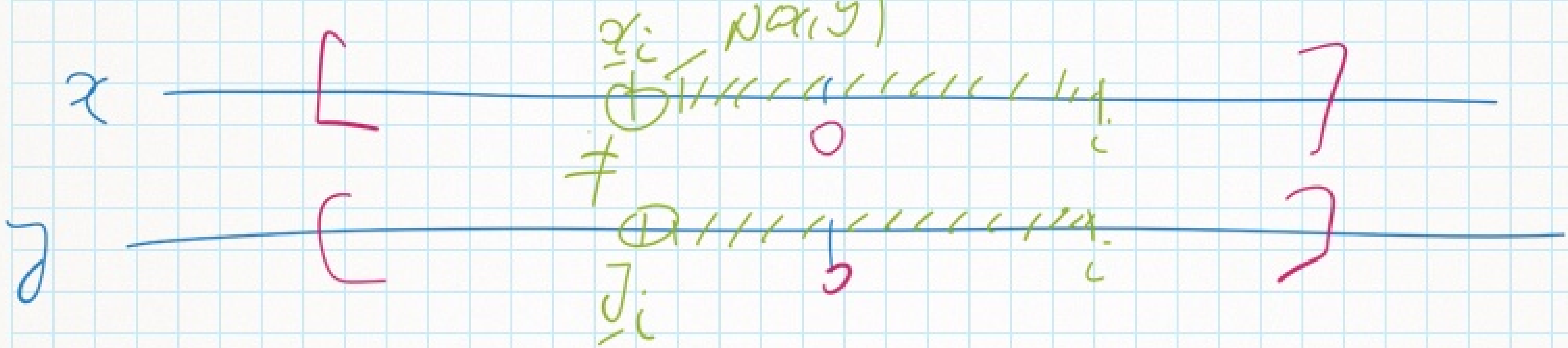
→ which topology in X?

We look $A^{\mathbb{Z}}$ as product space of a discrete set A with discrete topology, that is compact. So by Tychonoff $A^{\mathbb{Z}}$ compact.

space. This is metrizable.

$$d\Big(\underbrace{\overset{x}{\overbrace{(x_i)_{i\in\mathbb{Z}}}}}_{\in A^{\mathbb{Z}}},\ \underbrace{\overset{y}{\overbrace{(y_i)_{i\in\mathbb{Z}}}}}_{\in A^{\mathbb{Z}}}\Big) = \frac{1}{2^{N(x,y)}}$$

$$N(x,y) = \min\{\,|i| \mid i\in\mathbb{Z},\ x_i \neq y_i\,\}$$



So two points "are close" if they coincide in "big simmetric window around 0".
— coordinate.

For this metric: (X, T) is a top. dyn. syst.

compact metric

homeomorphism

$$\boxed{\text{For the moment "}\underline{\text{full shift}}\text{"} \to (A^{\mathbb{Z}}, T)}$$

<u>The notion of subshift</u>: $X \subseteq A^{\mathbb{Z}}$ is a subshift

iff X compact set, $\neq \emptyset$, $T(X) = X$ $\Leftarrow$ X is T-invariant

$\left(T|_X : X \to X \right.$ homeomorphism $\left. \right)$

<u>just T</u>

So a subshift define a top. dyn. system

$$\boxed{(X, T)}$$

Example 1:

① $X = \{ (x_i)_{i\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}} \mid \forall i \in \mathbb{Z},\ x_i x_{i+1} \neq 11 \}$

SFT

a way to look to X.

→ closed set for product topology (ex.)

→ clearly σ invariant.

② Fibonacci substitution subshift:

Fibonacci substitution $\sigma : \{a,b\} \longrightarrow \{a,b\}^+$ — concatenation of letters so words in a and b.

$a \longrightarrow \sigma(a) = ab$

$b \longrightarrow \sigma(b) = a$

$\sigma : A^* \longrightarrow A^*$, $A^* = A^+ \cup \{\varepsilon\}$ ← empty word.

This map can be extended to words:

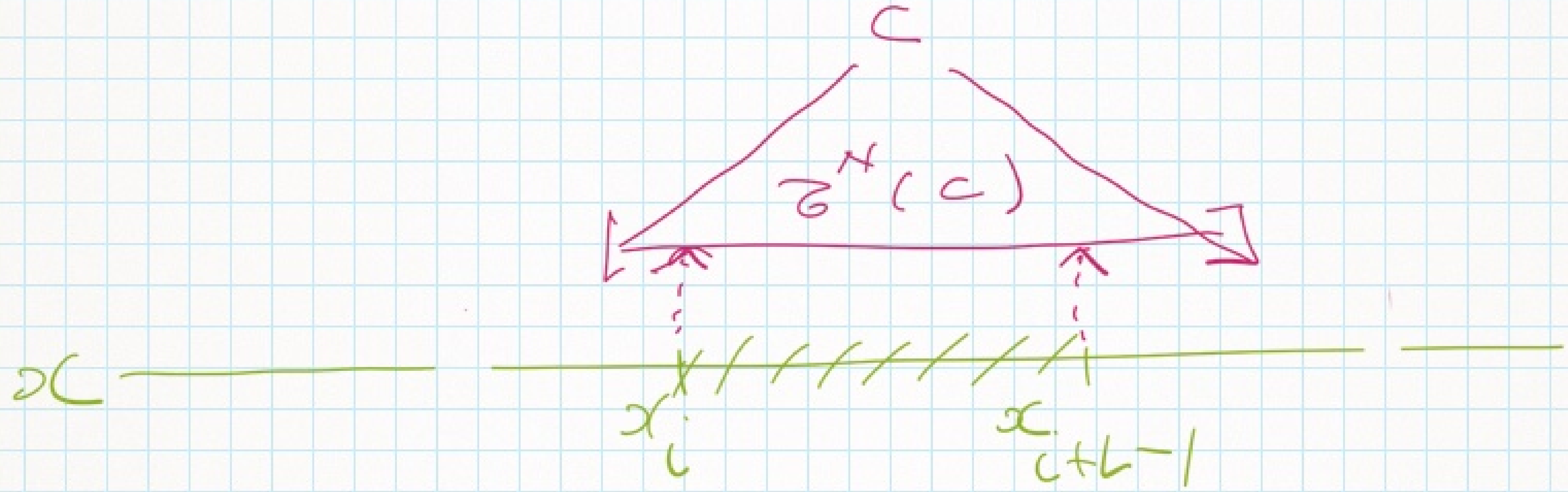
. $w = w_0 \cdots w_{L-1}$ with $w_i \in \{a,b\}$

$\sigma(w_0 \cdots w_{L-1}) = \sigma(w_0) . \sigma(w_1) \cdots \sigma(w_{L-1})$

It makes sense $\tau^l = \underbrace{\tau \circ \cdots \circ \tau}_{l\text{-times}}$.

$$X_\tau = \left\{ (x_i)_{i\in\mathbb{Z}} \in \{a,b\}^{\mathbb{Z}} \;\middle|\; \begin{array}{l} \forall i\in\mathbb{Z},\ \forall L \geq 0,\ \exists c \in \{a,b\},\ \exists N \geq 0, \\ x_i \cdots x_{i+L-1} \sqsubseteq \tau^N(c) \end{array} \right\}$$

$\sqsubseteq$ "to be a subword of"

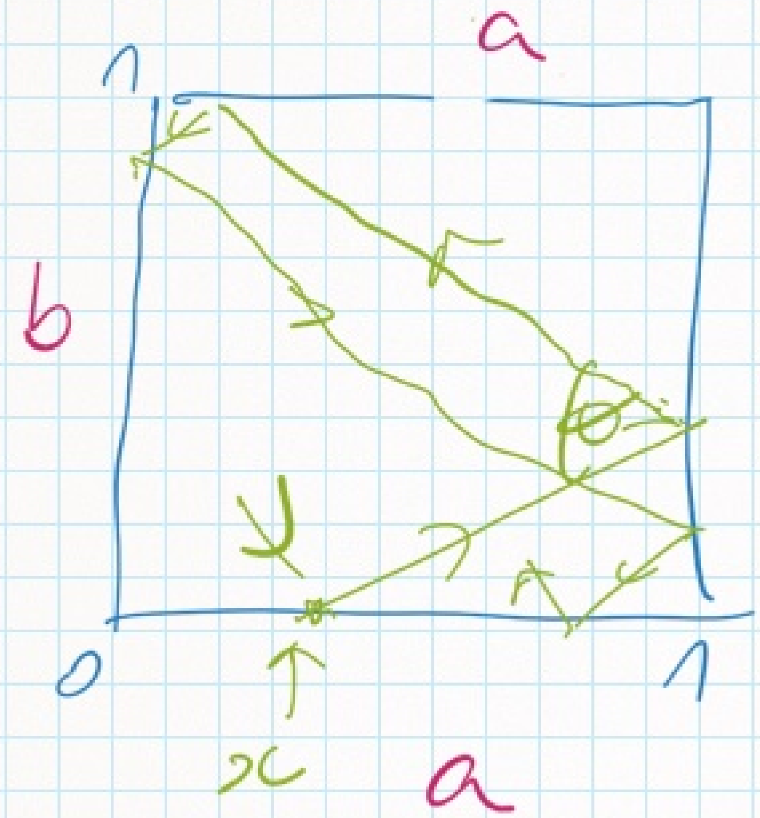


$\longrightarrow$ closed ...

$\longrightarrow$ T invariant ...

$\Rightarrow$ $\boxed{(X_\tau, T) \text{ is top. dyn. syst.}}$

This example can be constructed geometrically:



good choice of $\underline{x}$ and $\underline{\theta}$ produce a sequence $(a_i)_{i \in \mathbb{Z}} \in \{a, b\}^{\mathbb{Z}}$ that is in X_β.

Moreover, this sequence has dense orbit in X_β.

<u>Complexity of a subshift:</u> Given a subshift (X, T) the languages associated to X are:
$X \subseteq A^{\mathbb{Z}}$

$\mathcal{L}_n(X) = \{$finite words of length $\underline{n}$ appearing in points of $X\} \subseteq A^n$

Associated to such $\mathcal{L}_n(X)$ we have:

$$\boxed{p_X(n) = \# \mathcal{L}_n(X)}$$

complexity function

In previous examples:

SFT $\to$ $\# \mathcal{L}_n(X) \approx \left(\frac{1+\sqrt{5}}{2}\right)^n$

high complexity symb dynamics

$[X_2 \longrightarrow \# \mathcal{L}_n(X_6) = n+1$

low complexity symb. dynamics

Theorem (Hedlund-Morse): If (X, T) is a subshift s.t. $\exists N, \forall n \geq N, \quad p_X(n) \leq n$

$\Rightarrow X$ is finite.

[Exercise]

Another way to see that the examples are really different in terms of complexity is "entropy":

$$h_{top}(X,T) = \lim_{N\to\infty} \frac{1}{N} \log\left(P_X(N)\right)$$

topological entropy

invariant for conjugation

$$= \begin{cases} \frac{1+\sqrt{5}}{2} & \text{SFT} \\ \boxed{0 \qquad X_6} \end{cases}$$

We will live here in the next 2 talks.

Talk 2:

→ $X \subseteq A^{\mathbb{Z}}$, subshift if closed and shift invariant
$x = (x_i)_{i \in \mathbb{Z}}$ where $T: X \to X$
$(x_i)_{i \in \mathbb{Z}} \rightsquigarrow (x_{i+1})_{i \in \mathbb{Z}}$

→ $\mathcal{L}_n(X) =$ # words of length n appearing in points in X.

→ $\boxed{P_X(n) := \# \mathcal{L}_n(X)}$; $h_{top}(X,T) = \lim_{N \to \infty} \frac{1}{N} \log(P_X(N))$

"typically we will be looking to systems where $\boxed{h_{top}(X,T) = 0}$"

→ Hedlund-Morse: $\exists n \geq 1$, $\# \mathcal{L}_n(X) \leq n$
$\Rightarrow X$ is finite.

sketch proof: for the n s.t. $p_X(n) \leq n$

$(|p_X(n)| = n)$

$$\mathcal{L}_n(X) \ni \begin{array}{ll} w_1 & \underline{a_1} \\ w_2 & \underline{a_2} \\ w_3 & \underline{a_3} \\ w_n & \underline{a_n} \end{array}$$

$w_{j(n)}$

← continue in a unique way; so you produce a periodic configuration forward; the same can be said backward

$\boxed{\Sigma}$

So if X is infinite: $p_X(n) \geq n+1 \quad \forall n \geq 1$

The extreme case: $\boxed{p_X(n) = n+1 \quad \forall n \geq 1}$

name: sturmian subshift.

One example is the Fibonacci substitution subshift //

Substitutive subshifts:

Definition: Let A be a finite set and recall A^+ is the set of words in alphabet A and $A^* = A^+ \cup \{\varepsilon\}$
empty word.

→ The length of $\omega = \omega_0 \cdots \omega_{\ell-1} \in A^+$ is given by $|\omega| = \ell$

(non erasing)

→ A substitution is a morphism:

$$\sigma : A^+ \longrightarrow A^+$$
$$\omega \longmapsto \sigma(\omega)$$

$$\omega_0 \cdots \omega_{\ell-1} \Rightarrow \sigma(\omega_0 \cdots \omega_{\ell-1}) = \sigma(\omega_0) \cdots \sigma(\omega_{\ell-1})$$

So, to define σ is enough to know $\sigma(a)$ for $a \in A$.

Fibonacci	Morse:
$\sigma(a) = ab$	$\sigma(0) = 01$
$\sigma(b) = a$	$\sigma(1) = 10$
	...

→ We say that ζ is primitive if for $N > 1$, $\zeta^N(a)$ contains all the letters in A, this for all $a \in A$.

Another way to see the same is by defining a "matrix associated to ζ":

$$M_\zeta \in M_{A \times A}(\mathbb{N}) : \boxed{(M_\zeta)_{ab} = \# a\text{'s in } \zeta(b)}$$

In the examples:

Fibonacci: $M_\zeta = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

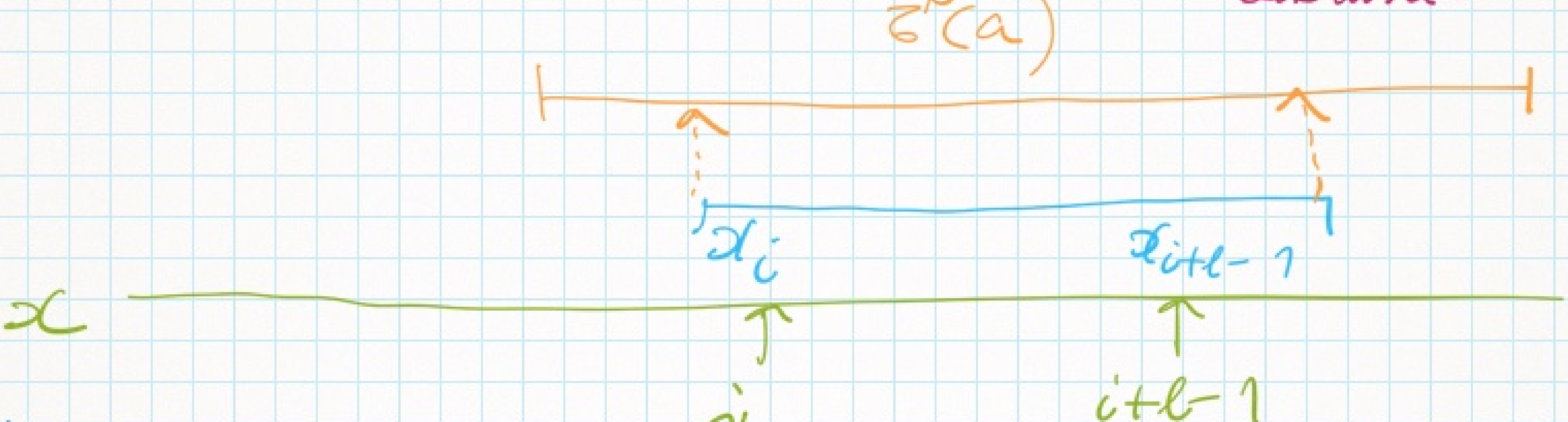
Morse : $M_\zeta = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

To be primitive means $\exists N \geq 1 : M_\zeta^N > 0$

From here we will only consider "<u>primitive substitutions</u>"

Given $\sigma : A^+ \longrightarrow A^+$ a p-subs. we define:

$$X_\sigma = \left\{ x = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}} \;\middle|\; \begin{array}{l} \forall i \in \mathbb{Z}, \forall \ell \geq 1, \exists a \in A, \exists N \geq 1 \\ x_i \cdots x_{i+\ell-1} \underset{\text{subword.}}{\sqsubseteq} \sigma^N(a) \end{array} \right\}$$

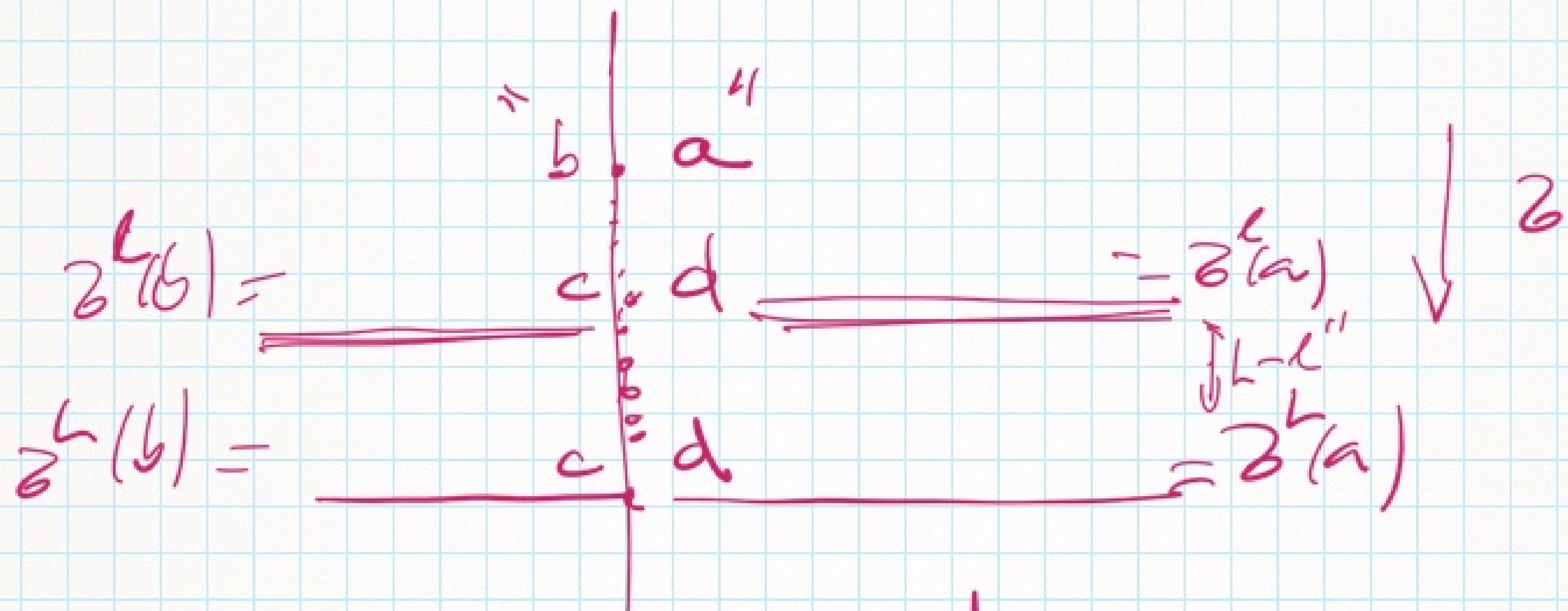


X_σ is clearly close and shift invariant.

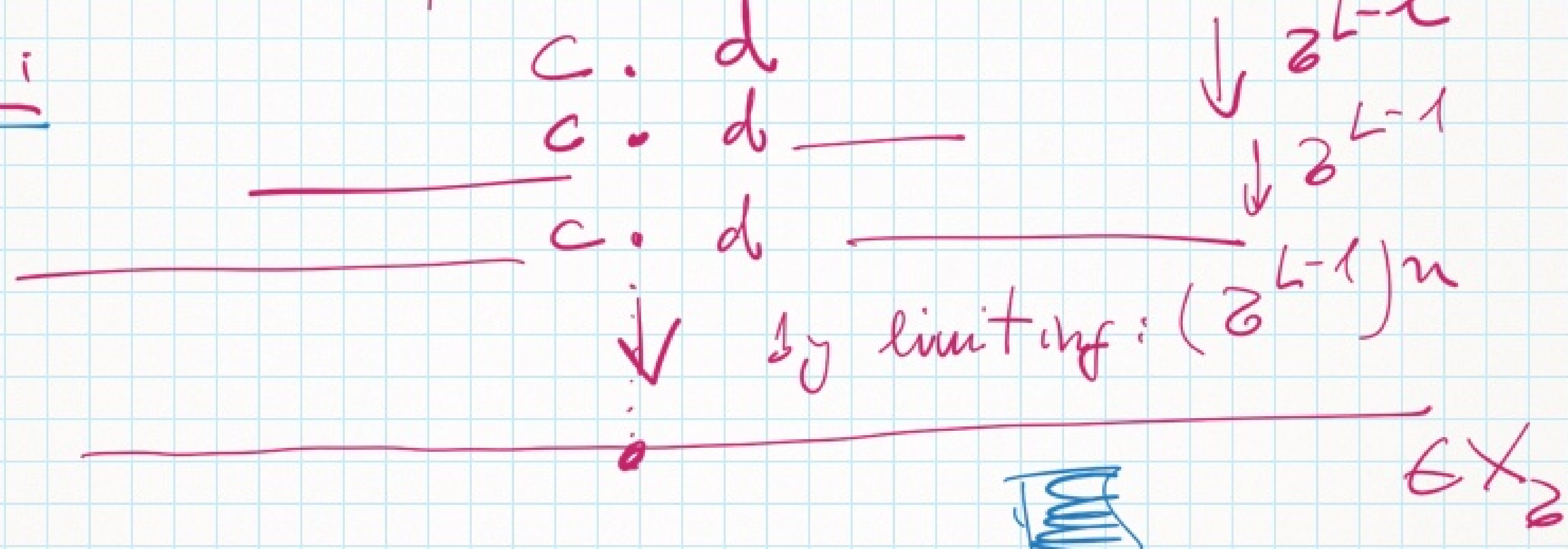
¿¿ is non-empty??

Intuitive proof of $\neq \emptyset$:

Consider $\exists\, a \in A^{\mathbb{Z}}$ appearing in some $\sigma^{**}(a)$,



Now:



Proposition: (X_σ, T) is minimal (all orbits are dense)

(σ is a primitive substitution).

$\longrightarrow$

Proof: what means for a subshift to be minimal?

$\forall x \in X_\sigma, \forall n \geq 1, \forall \omega \in \mathcal{L}_n(X_\sigma),$

$$\exists i \in \mathbb{Z}, \quad x_i \cdots x_{i+|\omega|-1} = \omega$$

• If $\omega \in \mathcal{L}_n(X_\sigma)$, $\exists a \in A$, $\exists N \geq 1$,

$$\omega \sqsubseteq \sigma^N(a)$$

• Also, we know that for every $\underline{i \in \mathbb{Z}}$ and $\underline{\ell \geq 1}$ "$|\omega|$"

$\exists b \in A, \exists M \geq 1, \quad \underline{x_i \cdots x_{i+\ell-1} \sqsubseteq \sigma^M(b)}$

since b appears in $\sigma^L(a)$ (by primitivity)
(any letter).

In particular:

$$x_i \dots x_{i+|w|-1} \sqsupseteq \sigma^{L+M}(a)$$

$$b$$

$$\downarrow$$

$$\dots\ a\ \dots\dots\ \sigma^{L}(b)$$

play L (as big as I want)

$$\downarrow$$

$$\dots\ \underbrace{\qquad}_{}\!\!\sigma^{N}(a)\ \dots \sqsupseteq w$$

x

Take L enough big s.t. $\boxed{\underset{\text{flexible}}{L} + \underset{\text{fix}}{N} = \underset{\text{big}}{M}}$

From here we will see w inside x ∎

Ex: fixing the proofs!! ?!?

other questions : 1) $\boxed{\text{weak-mixing ?}}$

2) mixing ?

3) $\boxed{\text{invariant measures.}}$

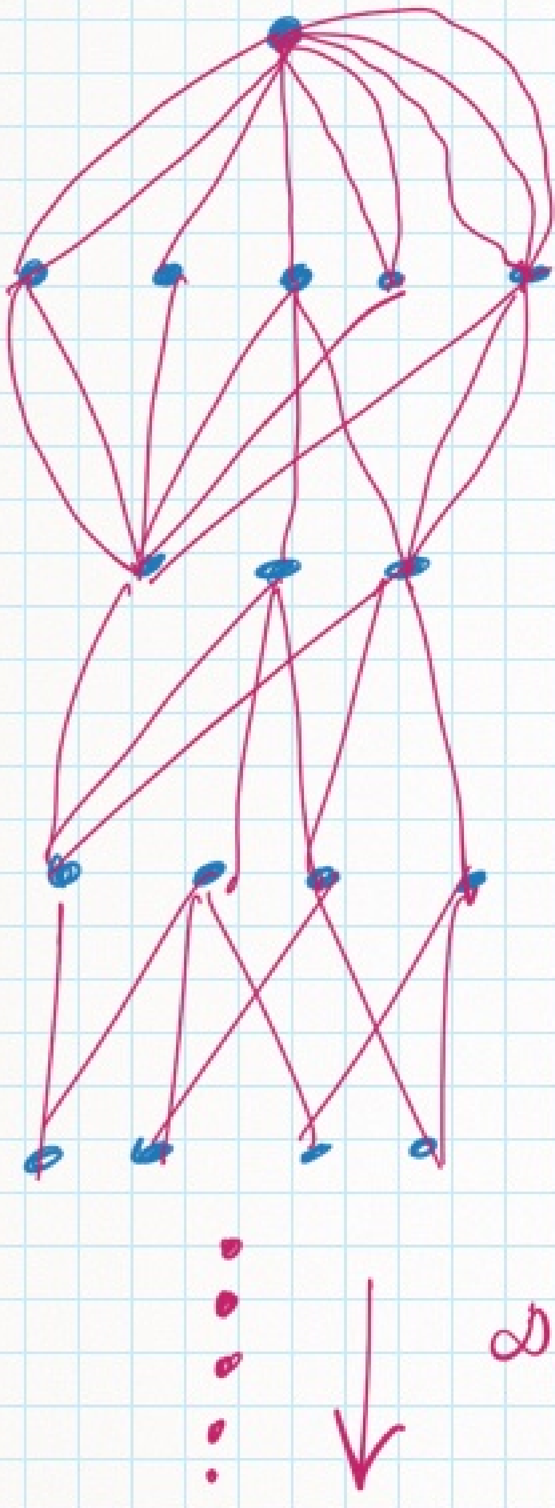
4) factors ...

(main references: works of Fabien Durand)

The way to see these questions will be via "Bratteli diagrams"

Bratteli-Vershik representations:

1) graph structure: (Bratteli diagram)



$V_0 = \{v_0\}$

$\leftarrow E_1$

V_1

$\leftarrow E_2$

V_2

$\leftarrow E_3$

V_3

$\leftarrow E_4$

V_4

∞'ly many times.

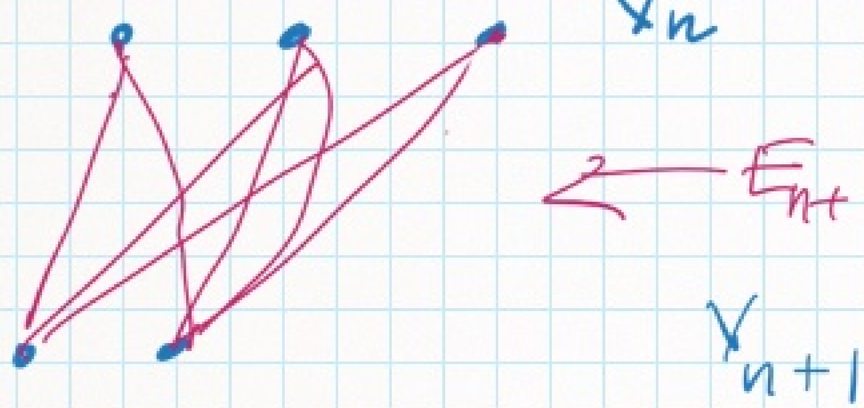
- The diagram is simple

$\forall n \geq 1 \; \exists m > n$ s.t. $\forall u \in V_n, \forall v \in V_m$ $\exists$ path of edges from u to v.

(consequence is that: to any vertex of level $n \geq 1$ arrives edges and starts edges

All our <u>Bratteli diagrams</u> will be <u>simple</u>

<u>Algebraic objects</u>:



$$M^{(n+1)} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$M^{(n+1)} \in M_{V_n \times V_{n+1}}(\mathbb{N})$$

$$M^{(n+1)}_{u,v} = \# \text{ edges between } u \in V_n \text{ and } v \in V_{n+1}$$

given <u>Bratteli diagram</u>: "$(M^{(n)} : n \geq 1)$"

obs. $M^{(1)}$ is a row vector: $M^{(1)} = [\, . \ . \ . \ . \ . \,] = H(1)$ hat of diagram

So we do the following:

$$\underset{V_0 \times V_1}{H(1)} \cdot \underset{V_1 \times V_2}{M^{(2)}} = \underset{V_0 \times V_2}{H(2)} = \begin{array}{c} v \in V_2 \\ \left[\quad \uparrow \quad \right] \end{array}$$

paths from v_0 to v.

$$\underbrace{\underset{V_0 \times V_1}{H(1)} \cdot \underset{V_1 \times V_2}{M^{(2)}} \cdot \underset{V_2 \times V_3}{M^{(3)}}} = \underset{V_0 \times V_3}{H(3)}$$

$\vdots$

$$\boxed{H(n) = H(1) \cdot M^{(2)} \cdots M^{(n)}} \in M_{V_0 \times V_n}(\mathbb{N})$$

$\longrightarrow \{ H(n) : n \geq 1 \} \Leftarrow$ heights of Bratteli diagram.

<u>simplicity</u> means that: $\forall n \geq \exists m > n$: "$M^{(n+1)} \cdots M^{(m)} > 0$"

2) Space:

$$X \subseteq E_1 \times E_2 \times E_3 \times \cdots$$

$$X = \left\{ (e_i)_{i\geq 1} \in \prod_{i\geq 1} E_i \;\middle|\; \forall i \geq 1,\ t(e_i) = s(e_{i+1}) \right\}$$

terminal vertex

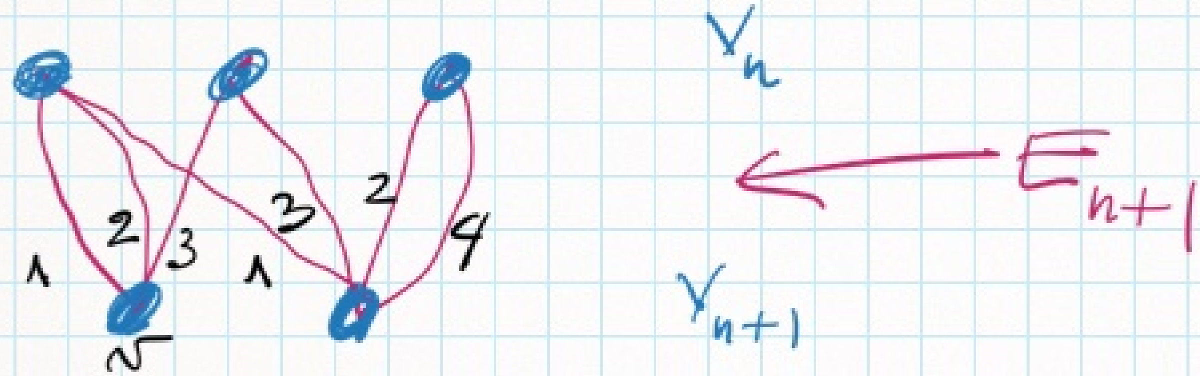
start vertex.

infinite paths starting in v_0 that we can draw in the Bratteli diagram.

For product topology of discrete spaces

X is compact, metri~~za~~ble.

3) A dynamics on the Bratteli diagram:



To each $n \geq 0$, $v \in V_{n+1}$, we order locally the edges $e \in E_{n+1}$ s.t. $t(e) = v$.

Crucial property: We say the local orders produce a Bratteli diagram that "properly order" if:

$\exists!\ x^- = (e_i^-)_{i \geq 1} \in X$: all e_i^- are minimal for local order

$\exists!\ x^+ = (e_i^+)_{i \geq 1} \in X$: all e_i^+ are max for local order.

From now we will be handling:

$$\boxed{\text{simple Bratteli diagrams that are properly ordered.}}$$

With those conditions we can define a dynamics on X:

first time e_ℓ is not max for local order

$$x = \overbrace{e_1 \; e_2 \; e_3 \cdots e_{\ell-1}}^{\text{max for local order}} \; \boxed{e_\ell}$$

$$+ \; 1 \qquad\qquad\qquad\qquad \downarrow +1$$

$$T(x) = \underbrace{f_1 \; f_2 \; f_3 \cdots\cdots f_{\ell-1}}_{\text{unique min for local order}} \; f_\ell \; e_{\ell+1} \cdots$$

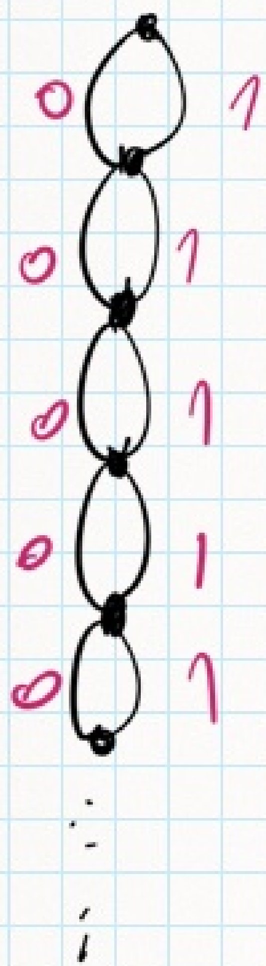
next edge in the local order of the edges arriving to $t(\underline{e_\ell}) = v$.

$$\boxed{T(x^+) = x^-}$$

$T : X \to X$ is a homeomorphism.

Examples:

1) odometer

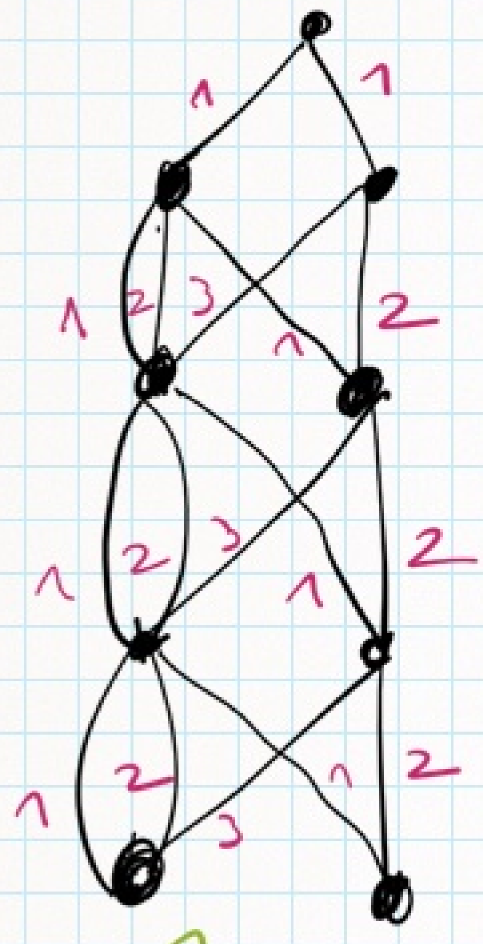


$$
\begin{array}{rl}
 & \;{}^{+1}\;{}^{+1}\;{}^{+1} \\
x = & 1\,1\,1\,0\,1\,1\,0\,0\,1\,1\,\ldots \\
+ & 1 \\
\hline
 & 0\,0\,0\,1\,1\,1\,0\,0\,1\,1\,\ldots \\
+ & 1 \\
\hline
 & 1\,\overset{+1}{0}\,0\,1\,1\,1\,0\,0\,1\,1\,\ldots \\
+ & 1 \\
\hline
 & 0\,1\,0\,1\,1\,1\,0\,0\,1\,1\,\ldots \\
+ & 1 \\
\hline
 & 1\,1\,0\,1\,1\,1\,0\,0\,1\,1
\end{array}
$$

"Dynamics on Cantor sets"

X

2)



$M^{(1)} = H^{(1)} = (1 \quad 1)$

$$M^{(2)} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$M^{(3)} = M^{(2)}$

$x = (1 \; 1 \; 1 \; 1 \; \ldots)$

$x^t = (1 \; 2 \; 2 \; 2 \; \ldots)$

"This diagram is simple and properly ordered"; so we can define $(X\ T)$ as before.–

"This kind of representations are called: Bratteli-Vershik systems"

substitution subshift

Theorem: "The Class
Substitutive subshifts
arising from "primitive substitutes"

is equivalent (conjugation)
to the class of Bratteli-Vershik systems
where all layer (up to the first one)
are identical (stationary
Bratteli Vershik
systems)
(when is not an odometer)

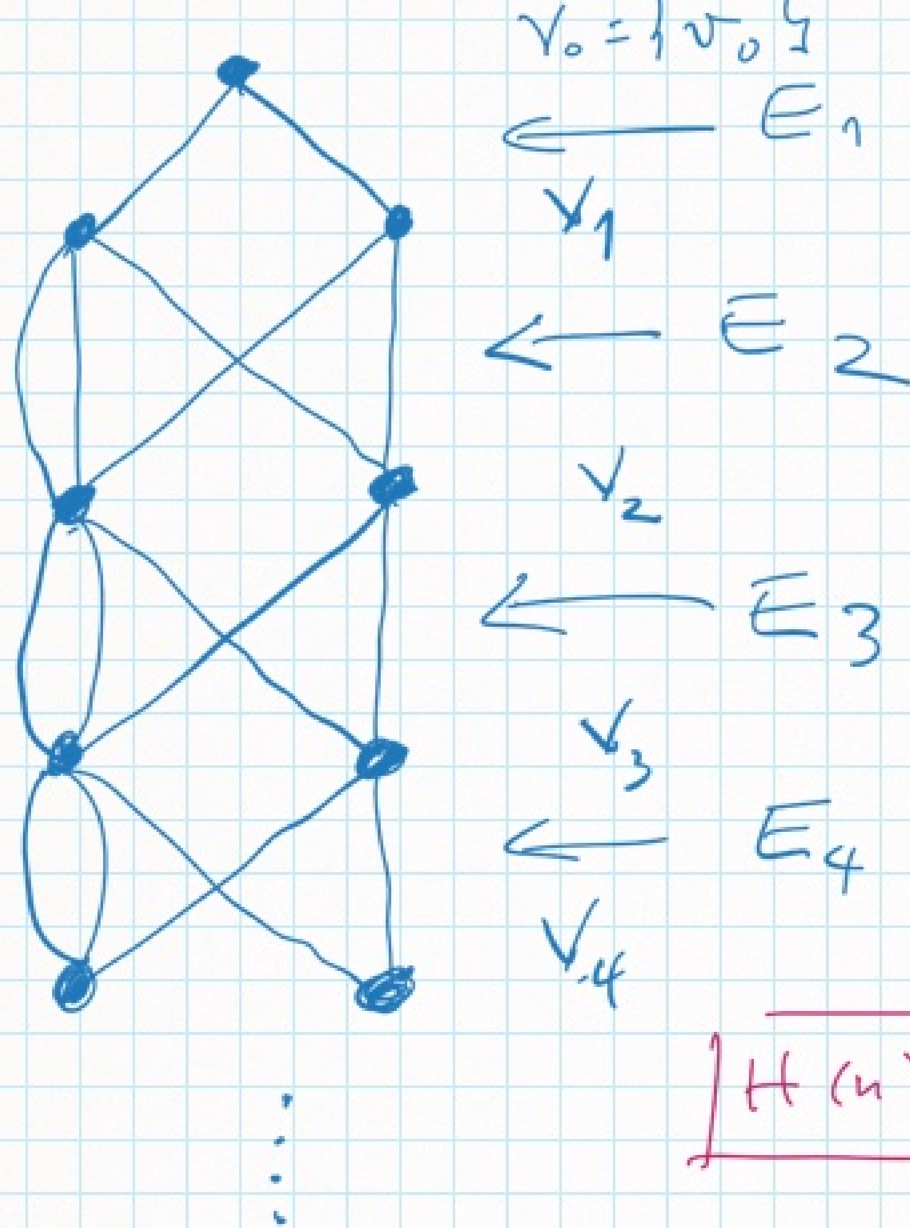
(Fabien Durand - B. Host - C. Skau).

Talk 3: we will prove unique ergodicity,
and study weak-mixing of
substitution subshifts using
such structures.

Talk 3:

Recall:

I



$$\begin{array}{cc} & v \in V_1 \\ M^{(1)} = M(1) = v_0 & \left[\ \ast\ \right] \end{array}$$

\# edges from v_0 to v

$M^{(2)} \in M_{V_1 \times V_2}(\mathbb{N})$

$M^{(3)} \in M_{V_2 \times V_3}(\mathbb{N})$

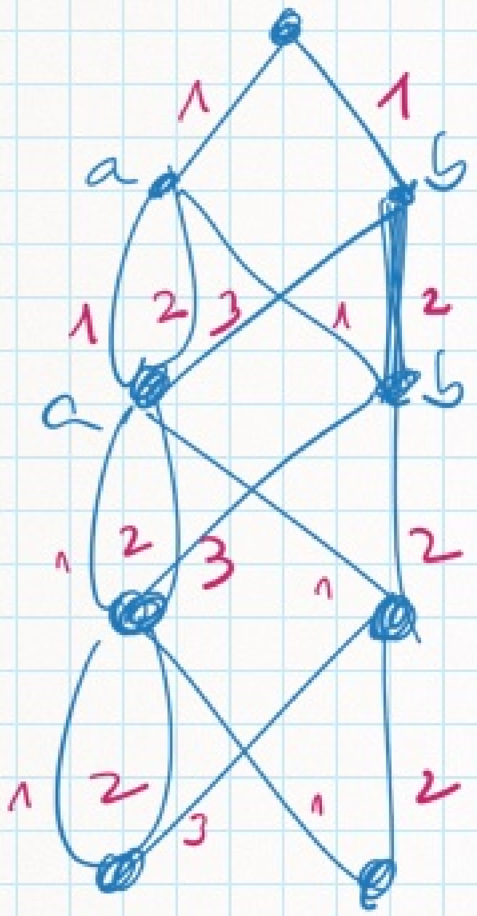
$\vdots$

$$\boxed{M(n) = M(1)\, M^{(2)} \cdots M^{(n)}}$$

Algebraic condition: "simple": $\forall n \geq 0,\ \exists m > n$,

" $P^{(n,m)} := M^{(n+1)} \cdots M^{(m)} > 0$ "

II order:



Dynamics:

first non maximum

$$x = \overbrace{e_1\ e_2 \cdots\cdots e_{\ell-1}}^{\text{all maximum}}\ \boxed{e_\ell}\ e_{\ell+1} \cdots$$

$$T(x) = \underbrace{f_1\ f_2 \cdots\cdots f_{\ell-1}}_{\text{all minimum}}\ f_\ell\ e_{\ell+1} \cdots$$

follower of e_ℓ in the local order

$$T(x^+) = x^-$$

X infinite paths from v_0.

T

Top. Dyn. System.

condition: "properly ordered"

$\exists!\ x^- = (e_i^-)_{i \geq 1}$, $\exists!\ x^+ = (e_i^+)_{i \geq 1}$

minimum for the local order

maximum for the local order.

Exercise: (X, T) is minimal.

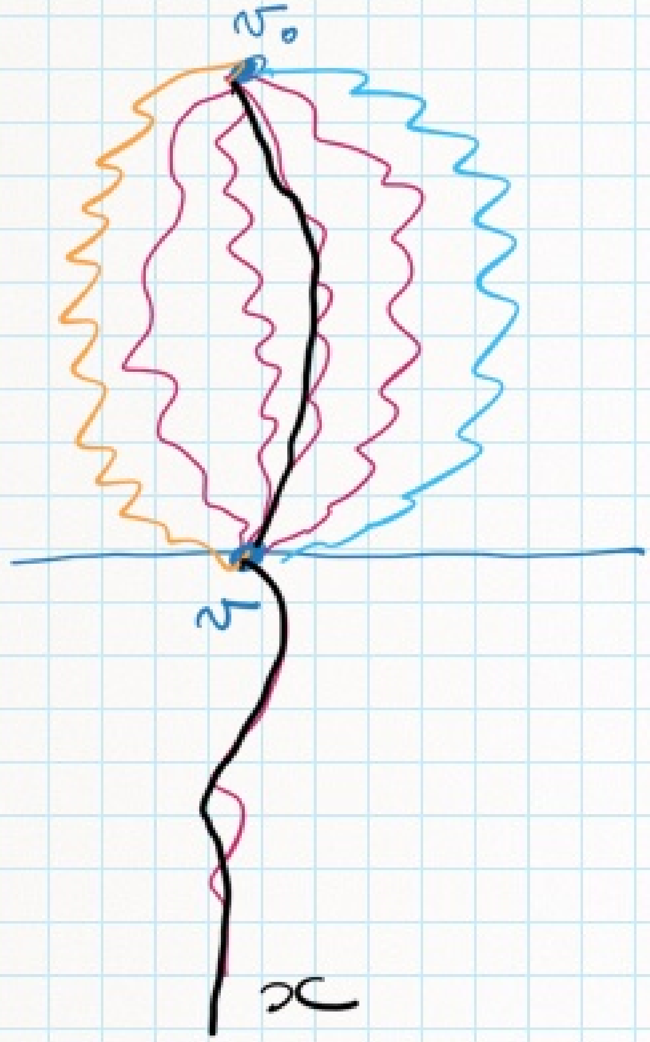
(we are assuming simplicity and proper order)

Theorem: stationary Bratteli–Vershik systems are "substitution subshifts" (with primitive substitutions).

In the example the substitution is:

$$\begin{aligned} \tau(a) &= aab \\ \tau(b) &= ab \end{aligned}$$

How to understand invariant measures:



- min
- max

(*) - T moves the path to covering all paths in between.

V_n

- each path $e_1 \dots e_n$ starting in v_0 and ending in $v \in V_n$ define a Borel set (closed and open set) of X:

$$[e_1 \dots e_n]_0$$

- By (*), if μ is an invariant prob. measure then:

$$(**) \quad \mu([e_1 \dots e_n]) = \mu([f_1 \dots f_n])$$

$\forall$ path joining v_0 and $v \in V_m$

Since the σ-algebra $\mathcal{B}_X$ (Borel one) is generated by

$$\boxed{\text{"}[e_1 \dots e_n]_0 \quad n \geq 1, \; e_1 \dots e_n \text{ a path from } v_0 \text{ to } v \in V_n\text{"}}$$

then μ is completely determined if we know all: $\mu([e_1 \dots e_n]_0)$. Since $(**)$ the measure μ is determined by vectors

$$\mu^{(n)} = (\mu_v [e_1 \dots e_n]_0 : v \in V_n)$$ column vector

any path from v_0 to $v \in V_n$

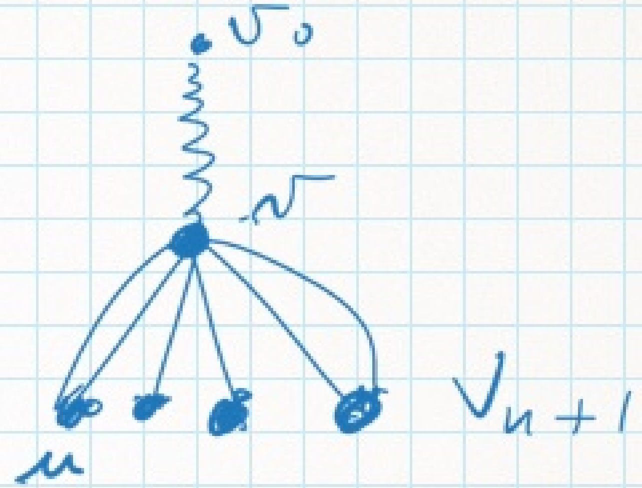
$$\mu^{(n)} = \begin{pmatrix} \vdots \\ \mu_v^{(n)} \\ \vdots \end{pmatrix} \; ; \quad \boxed{H(n) \cdot \mu^{(n)} = 1} \; (A_n)$$

values in $[0,1]$

Relation of $\mu^{(n)}$'s:

$\forall v \in V_n$:

$$\mu^{(n)}_v = \sum_{u \in V_{n+1}} M^{(n+1)}_{vu} \cdot \mu^{(n+1)}_u$$



$$\Rightarrow \ (\Delta_2) \ \boxed{\boxed{\underset{V_n \times 1}{\mu^{(n)}} = \underbrace{\underset{V_n \times V_{n+1}}{M^{(n+1)}} \cdot \underset{V_{n+1} \times 1}{\mu^{(n+1)}}}}}$$

(If the collection of $\mu^{(n)}$'s verifies (Δ)'s by extension Th. produce a unique prob. measure on $\mathcal{B}_X$)

By considering several stages:

$n \ll m$

$$\mu^{(n)} = \underbrace{M^{(n+1)} \cdots \cdots M^{(m-1)} M^{(m)}}_{P^{(n,m)}} \mu^{(m)}$$

Suppose: all $n^{(n)} = A > 0$ (case of substitution subshifts)

$\forall n < m$

$\Rightarrow$

$$\boxed{\mu^{(n)} = A^{m-n} \mu^{(m)}}$$

Perron-Frobenius Th.

$\forall n, \ \mu^{(n)} \in \langle \{ \overset{>0}{v_A} \} \rangle$

vector generating the eigenspace associated to λ_A, the P-F eigenvalue of A.

$\Rightarrow \mu^{(n)} = \alpha_n v_A$, but: $\quad H^{(n)} \cdot \mu^{(n)} = 1 \Rightarrow$ "$\alpha_n = \frac{1}{H^{(n)} \cdot v_A}$"

Clearly:

$$A\left(\alpha_{n+1}\cdot v_A\right)=\alpha_{n+1}\,\lambda_A\,v_A$$

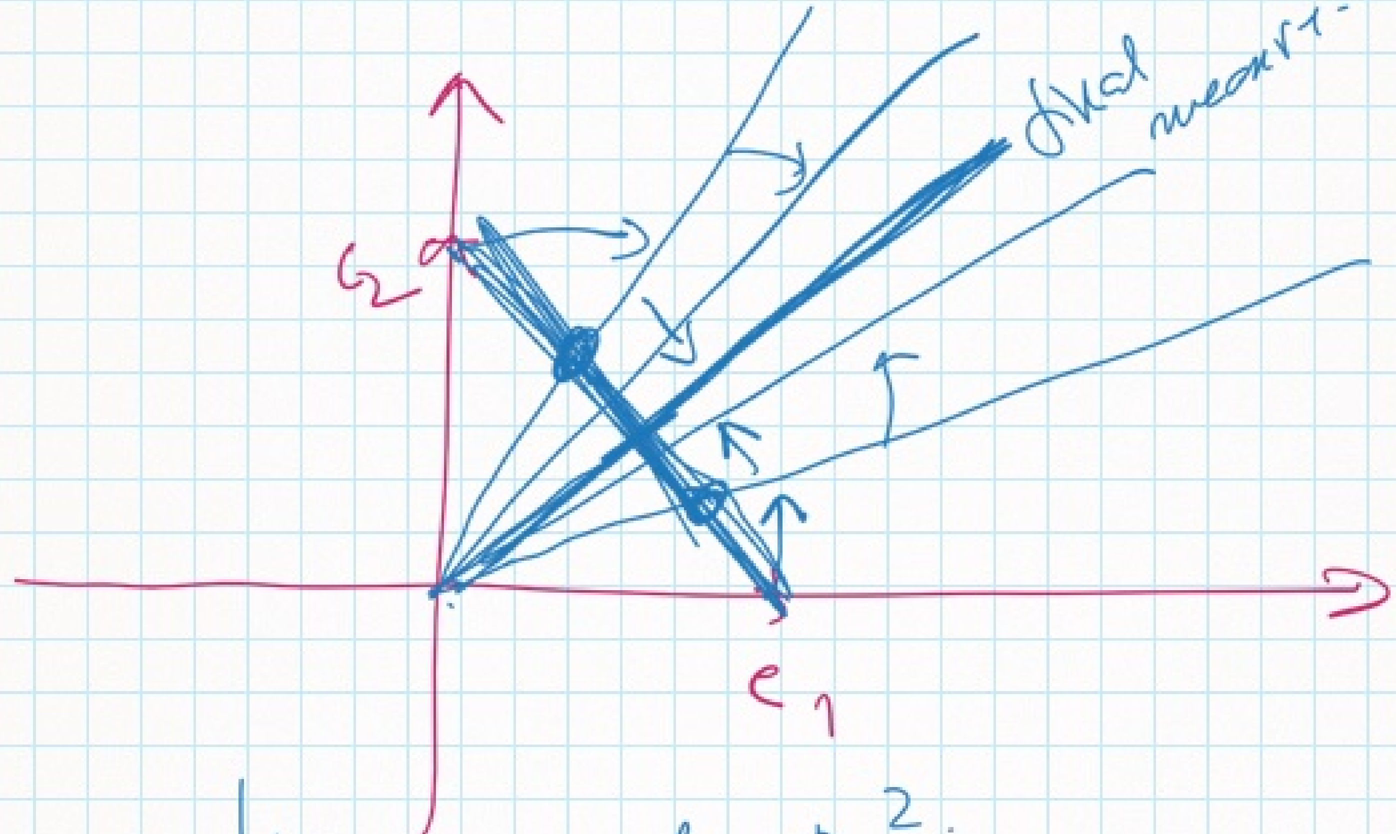
$$=\frac{1}{\underset{\underset{H(n)\cdot \underbrace{M^{(n+1)}\cdot v_A}_{\cancel{\lambda_A} v_A}}{\shortparallel}}{H(n+1)\cdot v_A}}\cdot \cancel{\lambda_A}\quad v_A$$

$$=\frac{1}{H(n)\cdot v_A}\cdot v_A=\alpha_n\,v_A$$

$$=\underline{\mu^{(n)}}$$

<u>Conclusion:</u> Any Bratteli–Vershik system (X,T) where $M^{(n)}=A>0$ $\forall n\geqslant 1$ is uniquely ergodic.

This is the case of "substitution on subshifts".



$K_{\mathbb{R}^2}$ positive cone of $\mathbb{R}^2$.

$\underbrace{\mu^{(n+1)} \cdot \cdot \cdot \cdot \cdot \mu^{(n+1)}}$ $\mu^{(n)}$

Exercise: Assume (X,T) is a Bratteli-Vershik system where "$n^{(n)}$" belongs to a finite set of >0 matrices.

Then, (X,T) is uniquely ergodic.

The systems like ↑ are called "Linearly Recurrent subshifts" (substitutions are here)

$X \subseteq A^{\mathbb{Z}}$ minimal subshifts s.t.

$\forall w \in \mathcal{L}(X)$, repetitions of w in points of X occur proportionally to the length of w, and this proportion is "universal".

$\exists K$:

x: ℓ w w

$\frac{1}{K}|w| \leq \ell \leq K|w|$

A general class of Bratteli–Vershik systems containing substitution subshifts and LR subshifts is the one of "topological finite rank systems":

(X,T) a Bratteli–Vershik system is top. F.R.

iff 1) $\forall n \geq 1,\ M^{(n)} > 0$;

2) $\# V_n = \ell \quad \forall n \geq 1$;

rank

Number of vertices is bounded $\# V_n \leq L \quad \forall n \geq 1$

Theorem: (Downarowicz/M.)

If (X,T) is top. F.R. then it is either a subshift or an odometer

(conjugate)

Th: (Durand–Donoso–M.–Petite)

If (X,T) is a subshift s.t. $\lim_{n\to\infty} \frac{p_X(n)}{n} < K$

$\Rightarrow$ is of top. FR

Comment: there are topFR systems that are not uniquely ergodic. In fact,

\# ergodic measures here is bounded by the rank of the system (ℓ)

~~"Top"~~

Weak-mixing property in top. finite rank systems

In this class, where all systems are minimal, WM means to do not have continuous eigenvalues

($f: X \to S^1$ continuous, s.t. $f \circ T = \lambda f$

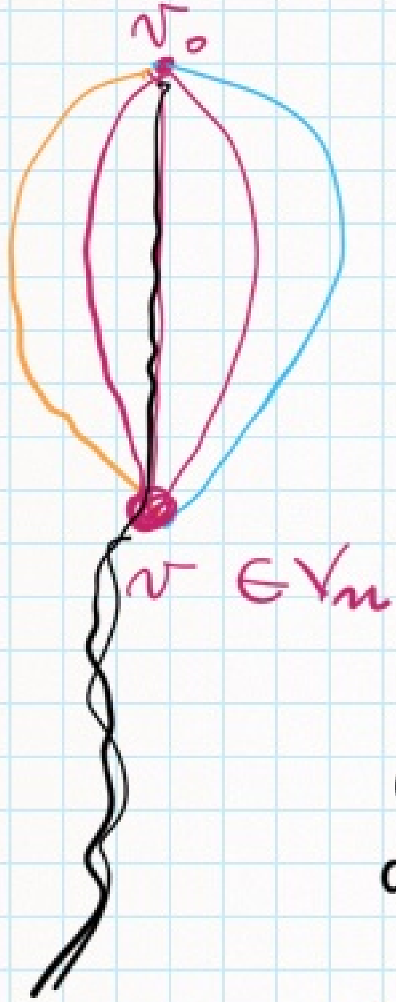
eigenvalue)

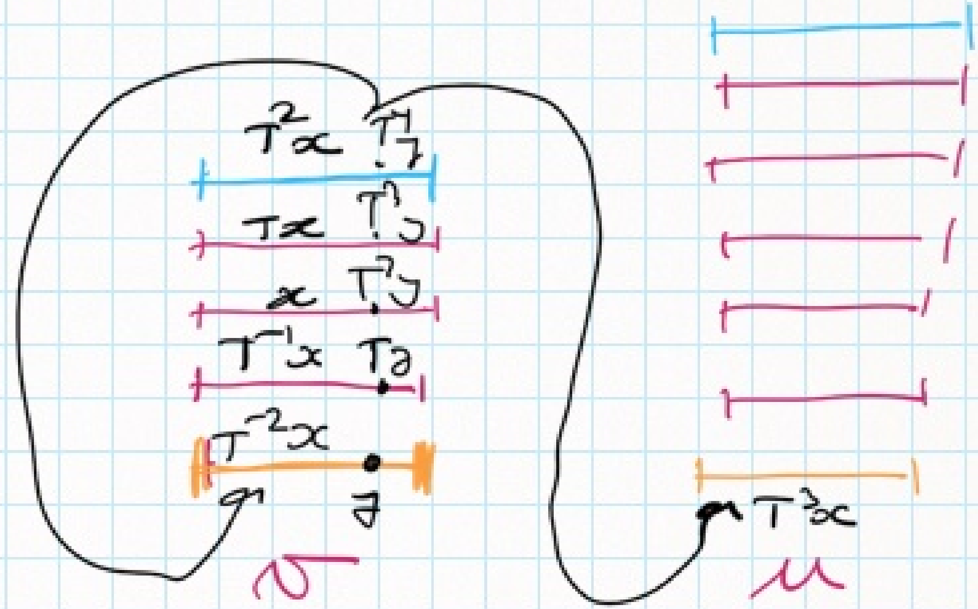
eigenfunction related to λ

$\lambda \in S^1 = \{ z \in \mathbb{C} / |z| = 1 \}$

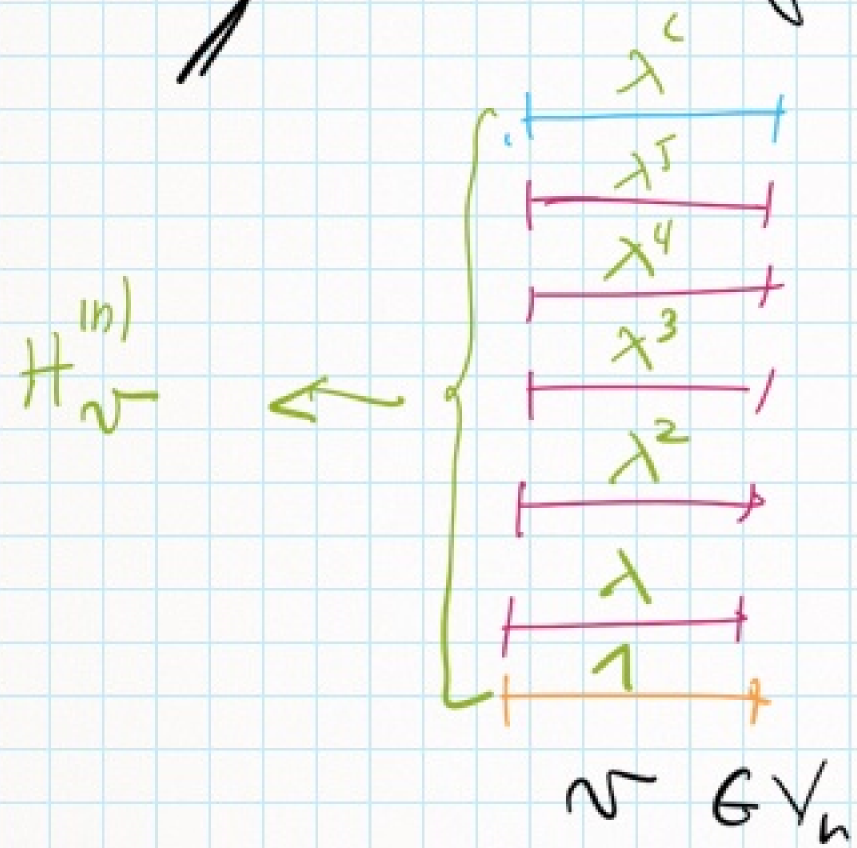
. .

how to address this problem:





Classical idea to construct an eigenfunction $f: X \to S^1$:



- $f_n^{(\lambda)}$ is constant at each stage of each tower:
- See if $f_n \xrightarrow[n\to\infty]{} f$ good topology of $C_{\mathbb{C}}(X)$

if true $f \circ T = \lambda f$.

to have a good approximation a condition that appears is that:

$$\lambda^{H_v^{(n)}} \xrightarrow[n\to\infty]{} 1 \qquad \forall v \in X_n$$

(if finite rank. $V_n = V$ $\forall n \geq 1$)

$\lambda = \exp(2i\pi\alpha)$

$$\alpha H_v^{(n)} \xrightarrow[n\to\infty]{} 0 \mod \mathbb{Z}$$

$\boxed{H^{(n)} A^n}$ ← study using Perron-Frobenius.

Theorems (Cortez, Durand, Host, M. / Bressaud–Durand–M. / Durand–Frank–M.)

Let (X,T) be a LR-subshift (given by a B-V system where $M^{(n)}$'s belong to a finite set of >0 matrices). Let μ be the unique invariant measure.

We have: $\lambda = \exp(2i\pi\alpha)$ is:

a continuous eigenvalue $\iff \sum_{n\geq 1} |||\alpha H^{(n)}|||^1 < \infty$

distance to integers

a "measurable" eigenvalue $(C_{\mathbb{C}}(X) \longrightarrow L^2_\mu(X)) \iff \sum_{n\geq 1} |||\alpha H^{(n)}|||^2 < \infty$

$\longrightarrow$ (weak-mixing for μ)

if: (X,T) is a substitution subshift we have $M^{(n)} = A > 0$ $\forall n \geq 1$.

$$H^{(n)} = H^{(1)} A^{n-1}$$

$\Rightarrow$ $|||\alpha H^{(1)} A^{n-1}|||$

$|||\alpha H^{(1)} A^{n-1}|||^2$ both go to 0 exponentially when this happens.

So "to be continuous or measurable eigenvalue is the same"

Th. (Host): $\longrightarrow$ many years before.–

TODAY: (Durand-Frank M.) We conditions for top FR subshifts, but more involved. use the orders //

→ LR: (Durand) $\Longrightarrow$ top. FR systems (B. Espinoza)
2022

⊙ finitely many symbolic factors

→ FRtop: (B. Espinoza, M.)

$\mathrm{Aut}(X,T)/\mathbb{Z}$ is finite

$\mathbb{Z} = \langle$ powers of shift $\rangle$

$\phi: X \longrightarrow X$ homeomorphisms

s.t. $\phi \circ T = T \circ \phi$

"END"

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